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DO COMPUTERS ENABLE MATHEMATICAL PROBLEM SOLVING OR JUST MAKE IT "EASY"?

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Abstract

This article suggests that recent advances in the development of computer programs capable of intricate symbolic computations might afford some negative outcomes by unintentionally shielding the learners of mathematics from all the challenges that are at the cornerstone of the subject matter. To address this concern, the paper describes how traditional curriculum can be modified in order to turn computer tools into enablers of problem solving. As an example of a new type of problem, the paper shows how the appropriate use of technology allows for some uncommon inquiries in the classic context of Fibonacci numbers. The article is written against the backdrop of mathematics teacher education.

Keywords: *teacher education, discovery experience, Fibonacci numbers, spreadsheets, Maple, Wolfram Alpha.*

1. INTRODUCTION

This article is informed by the author's more than two decades of participation in the preparation of teachers of mathematics for American and Canadian schools. The university where the author works is located in upstate New York, in close proximity to Ottawa, thus attracting Canadians to its master's degree programs in education. In this work, computer technology has been a critical component of mathematical learning environments, for the modern focus of mathematics teacher education research and practice is on the appropriate use of digital tools. As recommended by the Conference Board of the Mathematical Sciences — an umbrella organization consisting of seventeen professional societies in the United States (one goal of which is the improvement of mathematics teacher education) — "Teachers should become familiar with various software programs and technology platforms, learning how to use them to analyze data, to reduce computational overhead, to build computational models of mathematical objects and to perform mathematical experiments" [1, p. 57]. In what follows, the author's approach to addressing these recommendations of mathematicians concerned with the teaching of mathematics to teacher candidates in the digital era will be shared.

To begin, note that the advent of sophisticated computer programs in the contemporary educational environment that are capable of intricate symbolic computations has made many traditional problems from mathematics curricula kind of outdated as they can be solved by software almost at the push of a button. This is true for all levels of mathematics education — primary (e.g., doing rational number arithmetic), secondary (e.g., solving algebraic equations/inequalities), and tertiary (e.g., solving differential/difference equations). Moreover, many challenging problems turned out to be not immune from the effortlessness use of technology as well. Consequently, whereas technological advances have opened new research opportunities for professional mathematicians [2, 3], the widespread availability of powerful software tools has created challenges for mathematics educators involved in work with “mathematically proficient students” — a term used by the modern day educational document in the United States [4] to indicate that all students could and should develop mathematical proficiency, provided that their teachers are up to the task.

How can a student (including a teacher candidate) continue being motivated and keep up inquisitive spirit towards mathematics in the presence of technology? How can a computer be used as an enabler of problem solving rather than a shield against the challenges of learning mathematics, a shield that makes the subject matter not simply easier but “just easy”? In the 1980s, at the very outset of the digital era in education, the task of integrating technology at the advanced level of teaching mathematics was considered much more difficult one in comparison with the elementary level [5]. This was probably due to the fact that software tools available at that time were not sophisticated enough to enable their simple use in the teaching of advanced topics requiring high level knowledge of and skills in using programming languages [6, 7]. Nowadays, with significant decrease in the importance of those skills, more educational uses of technology that are markedly different from the practices of the past, are associated with conceptually rich themes of mathematics curricula. Computer algebra systems, spreadsheets, and dynamic geometry software are examples of the changing environment of the secondary mathematics classroom.

Nonetheless, new challenges for teaching advanced mathematics with technology may just be due to the availability of sophisticated computer programs capable of complex symbolic computations or graphic and geometric constructions. In terms of the theory of affordances [8] often used by mathematics educators when talking about technology [9, 10], the more positive affordances a tool offers, the fewer is the number of negative affordances it presents. In the context of mathematics, positive educational affordances of technology include new topics to be taught and the increase of the number of students who can comprehend advanced mathematical ideas. In other words, the use of technology has led to the emergence of new pieces of learnable mathematics [11]. Negative educational affordances of technology that are quite obvious include the reduction of mental computational skills and frequent emphasis on drill and practice leading to the development of automatism, something that often blocks creativity and insight [12]. However, negative affordances of a computational tool used in mathematics education may be hidden. For example, ready-made spreadsheet-based computational environments frequently require significant technical revision as a new version of the MS Office is released. Also, the availability of computer programs available free (or for a low cost) on-line reduces opportunities for a meaningful homework assessment by a teacher as it could be easily completed by a student with the help of software.

An old argument banning calculators from the mathematics classroom in order to uphold the tradition of the appreciation of mental computational skills can now be given a new attention at a higher level including algebra, trigonometry, calculus, and discrete mathematics. Thus, the use of technology in work with students of mathematics (including prospective teachers) is not a simple matter for its role is not to make doing mathematics just easier than before. In general, the appropriate use of technology in mathematics education can be conceptualized as a process that maximizes positive and minimizes negative affordances of a computational learning environment. One way to appropriately foster learners’ mathematical creativity and insight in the digital era is to present them with problems that, on one hand, are immune from the straightforward use of symbolic computations as a problem-solving method and, on the other hand, motivate and, most importantly, enable one’s creative thinking while solving problems.

2. TYPE II APPLICATION OF TECHNOLOGY OF THE SECOND ORDER

Consider *Wolfram Alpha* — a web-based computational engine developed by Wolfram Research (www.wolframalpha.com) available free on-line and, by accepting a natural language input, being capable of complex symbolic computations not possible (or two cumbersome) otherwise even for a professional mathematician. From the first glance, the program may be praised for enabling educators to bridge the gap between the past — when only some students were able to do mind-engaging mathematics, and the present — when an average student, without significant preparation in either mathematics or technology, is able to enjoy finding out a solution to a challenging problem using a computer. At the same time, by reducing problem solving to simply pushing buttons on a keyboard, the program can also put a barrier in the way of fostering creative skills in the students of mathematics. Once again, this dichotomy calls for the development of new instrumentation processes enabling the outcome of problem solving not to be dependent on what Guin & Trouche called "an *automatic transport phenomenon*" [13, p. 205, italics in the original] when one's ability of solving a mathematical problem is confined to simply entering correctly all data into a computer. A more difficult case is the use of a computer algebra system like *Maple* which does require certain level of programming skills for doing even basic symbolic computations. In many cases, the integration of a spreadsheet, *Maple*, and *Wolfram Alpha* makes it possible to pursue the following chain of mathematical activities: from conjecturing through numerical evidence (generated by a spreadsheet) to formal proof with symbolic computations supported by *Maple* and *Wolfram Alpha* which can complement each other in a variety of ways when carrying out complex algebraic transformations. As Langtangen and Tveito put it: "Much of the current focus on algebraically challenging, lengthy, error-prone paper and pencil work can be significantly reduced. The reason for such an evolution is that the computer is simply much better than humans on any theoretically phrased well-defined repetitive operation" [14, p. 811–812].

Nonetheless, one can use computer in a way that makes both numeric and symbolic computations more cognitively demanding despite or perhaps because of its automatic problem-solving capability. Maddux [15] introduced the notion of Type I/Type II technology applications seeing the latter type as "new and *better* ways of teaching" (p. 38, italics in the original). The growth in computational capabilities of mathematical software, enabling one to solve multistep problems at the push of a button, blurs the perceived dichotomy between the two types. To continue securing educational benefits from the Maddux's educational construct, the notion of Type II application of technology can be advanced to a higher level where one deals with what may be termed technology-immune/technology-enabled (TITE) problems [16]. Such problems cannot be automatically solved by software, yet the role of technology in dealing with them is critical. In that way, one can talk about TITE problem solving as a context for Type II application of technology of the second order.

For example, instead of posing a task, found in Pólya's famous book [17], to guess a rule for the sequence 11, 31, 41, 61, 71, 101, 131, ... (and, as shown in Figure 1, recognized by *Wolfram Alpha* as sequence A030430 from the On-line Encyclopedia of Integer Sequences (OEIS®; http://oeis.org/) — primes of the forms $10n+1$), the following questions can be asked: (i) What is the smallest number of divisions that one needs to decide whether the number 131 is a prime or not? (ii) How can one use technology to generate the first 100 primes of the form $10n+1$? (iii) What is the largest prime number in this sequence and how many divisions are required to decide whether this number is a prime or not?

Figure 1. Pólya's task via *Wolfram Alpha*

One can use a specifically designed spreadsheet environment [18] to carry out those divisions and generate prime numbers of the form $10n+1$. Yet, this commonly available tool, in the absence of mathematical reasoning and basic programming skills, does not give an immediate answer about the smallest number of divisions and makes the elimination of composite numbers from the sequence $10n+1$ a cumbersome process. Through answering question (i), one can also be introduced to the on-line sieve of Eratosthenes [19] and even complete all divisions using this tool. In doing so, one has to realize that after 131 had survived divisibility by 2, 3, 5, 7, and 11, no more division is necessary. Thus, the above modification of the task about the sequence of primes of the form $10n+1$ included in a classic book published more than 60 years ago may be considered a TITE problem.

In fact, elementary prime number theory, known, in general, as an excellent educational context for mathematically motivated students [20], can become the genesis of TITE problem solving. Learning to deal with TITE problems encourages computational experiment approach to school mathematics [21], an approach that can be traced back to the third century BC as evidenced by the following historically remarkable statement made by Archimedes in a letter to Eratosthenes, "Certain things first became clear to me by a mechanical method, although they had to be demonstrated by geometry afterwards because their investigation by the said mechanical method did not furnish an actual demonstration. But it is of course easier, when the method has previously given us some knowledge of the questions, to supply the proof than it is to find it without any previous knowledge" [22, p. 13]. In the modern terms, Archimedes, who is considered among the greatest mathematicians of all time [23], had acknowledged the usefulness of, "at least initially, approaching the mathematics from an experientially based direction, rather than an abstract/deductive one" [24, p. 96]. In a TITE problem-solving context, a computer enables initial understanding of a problem at hand by offering "a mechanical method", something that helps one to move to the plane of abstract thought in order to complete a solution.

3. TITE PROBLEM SOLVING AS A PRECURSOR OF MATHEMATICS RESEARCH

While the above perspective on the revision of mathematics curriculum in the digital era is not limited to a specific educational context, using TITE problems can contribute to the development of creative thinking of mathematically proficient students and their teachers alike. The teachers can learn posing TITE problems for the students by modifying/expanding already existing challenging (non-traditional) problems through the use of technology. The students can learn dealing with TITE problems, including teacher-posed and self-posed ones, in an exploratory fashion. Explorations of that kind are consistent with the modern day approach to mathematical research when (both old and new) problems become solvable only due to the capability of computers to carry out symbolic computations not possible otherwise [25].

It should be noted that, unlike experimentation in mathematics, computational experiment in mathematics education usually does not offer results that were not possible to obtain in the pre-digital era. Yet, the teaching ideas shared in the next section, although having mostly educational merit, enable the demonstration of how the appropriate use of technology in mathematics education can open a window to new ways of inquiry into a classic concept of mathematics and contribute to the body of knowledge about such a concept. In particular, from this demonstration one can learn how numeric sequences can be described through difference equations — mathematical models of discrete dynamical systems used in important applications to science and engineering — thereby providing a new perspective on STEM (science, technology, engineering, mathematics) education.

4. CONSTRUCTING (p, k)-SECTION OF FIBONACCI NUMBERS AS A TITE PROBLEM

Consider the following well-known number sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \dots, \tag{1}$$

in which every number beginning from the third is the sum of the two preceding numbers and the first two numbers are equal to one. Changing the first two terms of sequence (1) but preserving the rule through which other terms develop, yields

$$2, 1, 3, 4, 7, 11, 18, 29, 47, \dots. \tag{2}$$

Sequence (2) is named after Lucas [26] who, according to Koshy [27], gave sequence (1) its celebrated name, Fibonacci numbers. Both sequences are described through the same difference equation, $f_{n+1} = f_n + f_{n-1}$ varying only in the initial values, f_0 and f_1. Such sequences are called Fibonacci-like sequences.

Through unsophisticated spreadsheet-based experimentation with the first two terms of sequences (1) and (2), one can discover that, regardless of f_0 and f_1, the ratios f_{n+1}/f_n always tend to the Golden Ratio $\phi = 1.61803$, as n increases. Finding the exact value of $\phi = \frac{1+\sqrt{5}}{2}$, without googling the words “golden ratio”, requires knowledge of Binet’s formula

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right], n = 0, 1, 2, \dots, \tag{3}$$

for sequence (1), the formal derivation of which is quite complicated and it can be replaced by typing the words “Binet’s formula for Fibonacci numbers” into the input box of *Wolfram Alpha*) Assuming a certain level of mathematical proficiency on the part of a problem solver, finding the exact value of ϕ can be considered a TITE problem – first enabling the “discovery” of formula (3) using *Wolfram Alpha* and then finding $\lim_{n\to\infty} \frac{F_{n+1}}{F_n}$ with paper and pencil only. Similarly, using dynamic geometry software (e.g., The Geometer’s Sketchpad), one can first discover the presence of ϕ in a regular pentagon experimentally and then confirm the discovery through a formal demonstration by combining the machinery of trigonometry and complex numbers.

A more sophisticated (TITE oriented) exploration that can be introduced in the context of Fibonacci and Lucas numbers is what may be called (p, k)-section of sequence (1) — the process of elimination of all Fibonacci numbers the ranks of which are not divisible by p, then doing the same elimination for the surviving sequence, and so on, k times. The terms of the surviving (after k eliminations) subsequence of sequence (1) will be denoted $f_{p^k n}$. Note that the case $p = 2$ was referred to in [28] as Fibonacci sieve of order k. In particular, Fibonacci sieve of order 1 (that is, the sequence 1, 2, 5, 13, 34, 89, ...) is referred to as a bisection of Fibonacci sequence and numbered A001519 in the OEIS® thus explaining the term (p, k)-section used in this paper. However, already Fibonacci sieve of order 3 (i.e., the (2, 3)-section of Fibonacci numbers) is not included in the OEIS®, let alone the general case.

To proceed further, consider the case $p = 3$. Assuming that the first term of sequence (1) has rank zero, the sequence

$$1, 3, 13, 55, 233, 987, 4181, 17711, 75025, 317811, \ldots \tag{4}$$

survives the suggested process of elimination (trisection of the first order of Fibonacci numbers), where the first and the second terms of sequence (4) are Fibonacci numbers of ranks zero and three, respectively. Note that the terms f_{3n} of sequence (4) are Fibonacci numbers the ranks of which are consecutive multiples of three. One can use a spreadsheet to develop this sequence electronically (such use of a spreadsheet is itself a TITE problem). Then, one can be asked to find a difference equation that generates the sequence f_{3n}. By trial and error, one can come up with the difference equation

$$f_{3(n+1)} = 4 f_{3n} + f_{3(n-1)}, \tag{5}$$

where $f_0 = F_0, f_3 = F_3, n = 1, 2, 3, \ldots$.

Once again, using a spreadsheet, one can verify that difference equation (5) indeed generates sequence (4) and then use either *Wolfram Alpha* or *Maple* (or both) to obtain Binet's formula for the sequence f_{3n} in the form

$$f_{3n} = \frac{1}{2\sqrt{5}} \left[(\sqrt{5}+1)(2+\sqrt{5})^n + (\sqrt{5}-1)(2-\sqrt{5})^n \right], n = 0, 1, 2, \ldots$$

As an aside, note that formal derivation of Binet's formulas for the sequences f_n and f_{3n} can be used in a grade appropriate context to demonstrate an analogy with the solutions of linear differential equations that are developed through the reduction of the corresponding system to the Jordanian form followed by the return to the original phase space. A pedagogical importance of making this observation in the context of mathematics teacher education is to help teacher candidates "make insightful connections between the advanced mathematics they are learning and high school mathematics they will be teaching" [24, p. 39]. In doing so, one gravitates away from the candidates' developing what Cuoco [29] has called "vertical disconnect" — not seeing the connection between ideas taught in undergraduate mathematics courses and the school mathematics curriculum.

Continuing in the same vein, the next step is to apply the trisection process to sequence (4) that yields the sequence

$$1, 55, 4181, 317811, \ldots, \tag{6}$$

which survives the process of elimination (trisection of the second order) where the first two terms are, respectively, Fibonacci numbers of ranks zero and nine. Assuming that sequence (6) develops similar to sequence (4), on can construct the equation $4181 = 55x + 1$ to get $x = 76$. This suggests that sequence (6) satisfies the difference equation

$$f_{9(n+1)} = 76 f_{9n} + f_{9(n-1)}, \tag{7}$$

where $f_0 = F_0, f_9 = F_9, n = 1, 2, 3 \ldots$.

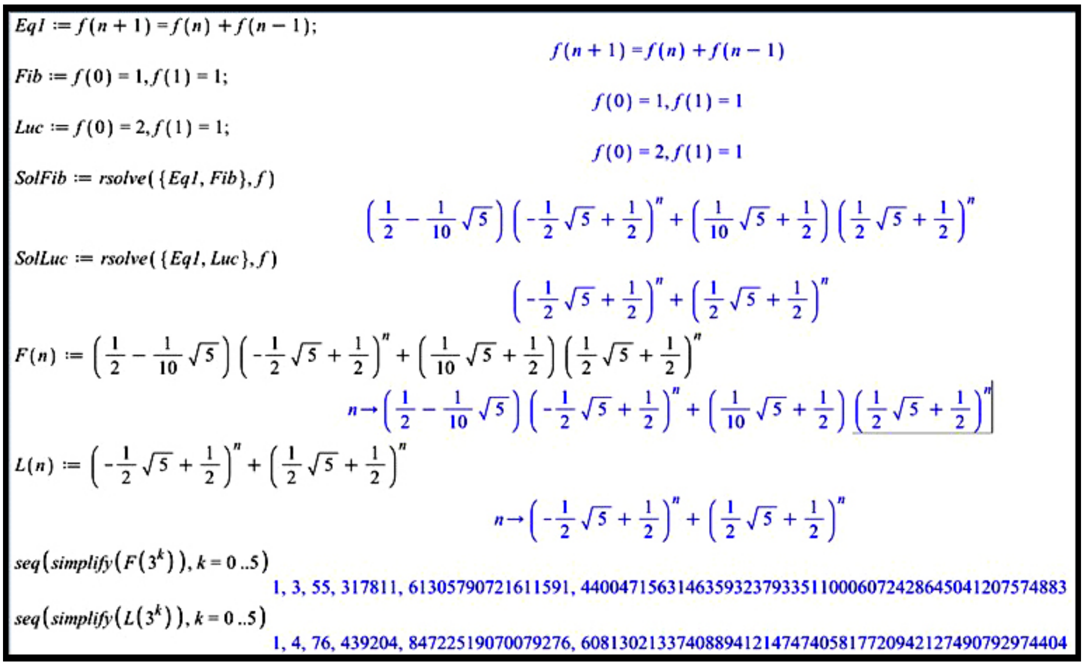

Figure 2. Using *Maple* to obtain Binet’s formulas and compute L_{3^k}

Whereas technology can be used to obtain Binet’s formula for the sequence f_{9n}, a more interesting question is about the origin of the coefficients in difference equations (5) and (7). As shown in [28], the process of consecutive bisection of Fibonacci numbers leads to similar difference equations the coefficients of which are Lucas numbers the ranks of which are the powers of two. This prompts one to generate a sufficiently large set of Lucas numbers to recognize that coefficients 4 and 76 in difference equations (5) and (7) are Lucas numbers of ranks 3 and 9, respectively. One can further use *Maple* to verify that the relation $L_{3^k} f_{n3^k} = f_{(n+1)3^k} - f_{n3^k}$ holds true for $k = 3, 4, 5$ (Figure 2). The above numerical evidence can prompt the formulation of

Proposition 1. The family of difference equations

$$f_{3^k(n+1)} = L_{3^k} f_{3^k n} + f_{3^k(n-1)}, \tag{8}$$

where $k, n = 1, 2, 3, \ldots,\ f_0 = F_0,\ f_{3^k} = F_{3^k}$ describes the $(3, k)$-section of sequence (1).

Proof. Consider Binet’s formula for Lucas numbers

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n, \tag{9}$$

which, just as formula (3), can be provided by technology (e.g., by *Maple* as shown in Figure 2). Let $\lambda_1 = \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{1-\sqrt{5}}{2}$. Then, the use of formulas (3) an (9) yields the following chain of equalities that prove relation (8):

$$\begin{aligned}
&\sqrt{5} L_{3^k} f_{n3^k} = (\lambda_1^{3^k} + \lambda_2^{3^k})(\lambda_1^{n3^k+1} + \lambda_2^{n3^k+1}) = \\
&= \lambda_1^{(n+1)3^k+1} - \lambda_2^{(n+1)3^k+1} + (\lambda_1\lambda_2)^{3^k}(\lambda_1^{(n-1)3^k+1} - \lambda_2^{(n-1)3^k+1}) = \\
&= \sqrt{5} f_{3^k(n+1)} + (-1)^3 \sqrt{5} f_{3^k(n-1)} = \sqrt{5}(f_{p^k(n+1)} - f_{p^k(n-1)}).
\end{aligned}$$

Proposition 1 can be immediately generalized by replacing a $(3, k)$-section by a (p, k)-section and then proved by replacing 3 by p in the above proof.

Proposition 2. The family of difference equations

$$f_{p^k(n+1)} = L_{p^k} f_{p^k n} + (-1)^{p+1} f_{p^k(n-1)}, \quad f_0 = F_0, \quad f_{p^k} = F_{p^k}, \tag{10}$$

where $k, n = 1, 2, 3, \dots$ describe the (p, k)-section of Fibonacci number sequence (1).

Remark. Proposition 2 holds true not only for Fibonacci numbers but also for any Fibonacci-like sequence. For example, taking the (3, 2)-section of the sequence 3, 1, 4, 5, 9, 14, 23,... yields the sequence 3, 97, 7375, 560597, 42612747, 32391293369, ..., which satisfies difference equation (10) where $p = 3, k = 2, f_0 = 2, f_9 = 97$ (e.g., one can check to see that). The fact that this sequence is not included in OEIS® may serve to teacher candidates (and other populations of students alike) as a motivation for generating new sequences of that kind by experimenting with different Fibonacci-like sequences.

5. CONCLUSION

The intent of this article was to argue that computational power of the modern tools of technology should not be taken to mean that students are finally provided with a short and easy path towards the answer to a mathematical question. Such naïve perspective on the use of modern technology has real potential to reduce all the challenges of the subject matter to simply pushing the right buttons on a keyboard. With this in mind, a didactic idea of the technology-immune/technology-enabled (TITE) problem-solving curriculum was presented and discussed.

The idea was further illustrated using a classic context of Fibonacci numbers within which the notion of a (p, k)-section of a Fibonacci-like sequence was developed. It was shown how using intuition, trial and error, multiple tools of technology, and, most importantly, natural curiosity and drive for discovery, one can construct the family of difference equations depending on Lucas numbers as a formal description of the (p, k)-sections. Whereas proofs of relations (8) and (10) are almost identical and may even be considered "easy", it seems unlikely that without numerical evidence provided by technology one can come up with those relations. It is a computational experiment that brought them into existence. The approach of enabling mathematical exploration through the use of technology was shown having ancient roots as one can describe the content of section 4 with a reference to the above-cited statement by Archimedes. Indeed, without a "mechanical method" made possible by the application of multiple computer tools — a spreadsheet, *Wolfram Alpha*, and *Maple* — the notion of the (p, k)-section of sequence (1) and connection between Fibonacci and Lucas numbers through constructing difference equations (7) and (9) may not come to light.

The notion of a TITE problem-solving curriculum was presented, in part, as an extension of the Type II application of technology educational construct to allow for seeing its merit for mathematics education even in the era of highly sophisticated digital tools. It was suggested that an experimental approach to the teaching of mathematics, including the education of teachers, can help teacher candidates to better appreciate a true meaning of mathematical discovery. Teachers, as Pólya [30] once advised, cannot impart the experience of mathematical discovery to their students if they themselves had not have this experience. It appears that using technology as an enabler of TITE problem solving can do away with seeing educational computing through the "math is easy" lens, and, instead, can lead to the growth of mathematical proficiency of students and discovery experience of their teachers.

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РОЛЬ КОМПЬЮТЕРА В МАТЕМАТИЧЕСКОМ ОБРАЗОВАНИИ — ДУМАТЬ ВМЕСТЕ СО СТУДЕНТОМ ИЛИ ДУМАТЬ ЗА СТУДЕНТА?

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Аннотация

Новейшие достижения в разработке компьютерных программ способных производить сложные символические вычисления могут иметь нежелательные педагогические последствия. Простота вычислений способна непреднамеренно оградить учащихся от трудностей связанных с решением задач — основой основ математического образования. В этой связи, обсуждается идея модификации учебных материалов с тем, чтобы компьютер мог бы служить педагогически правильным помощником в решении задач разного уровня. В качестве примера нового типа задач предлагается не обсуждавшееся ранее компьютеризированное исследование в классическом контексте чисел Фибоначчи. Статья написана по опыту преподавания математики будущим учителям средней школы.

Ключевые слова: *подготовка учителей, опыт открытия, числа Фибоначчи, спредшит, Мэйпл, Вольфрам Альфа.*

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